Notes 7.4 Determinants and Cramer's Rule

Solve systems of equations using:

- The determinant to decide if a Matrix has an Inverse
- Cramer's rule to solve a system of equations

Determinant of a 2 X 2 Matrix

1-8 Find the determinant of the matrix, if it exists.

4. $\begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix}$

6. $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

The **minor** M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A.

The ***cofactor*** A_{ij} of the element a_{ij} is

$$A_{ij} = (-1)^{i+j} M_{ij}$$

To illustrate these definitions, consider the following 3 by 3 matrix,

$$\begin{bmatrix} 1 & 4 & 7 \\ 3 & 0 & 5 \\ -1 & 9 & 11 \end{bmatrix} \qquad M_{2,3} = \det \begin{bmatrix} 1 & 4 & \square \\ \square & \square & \square \\ -1 & 9 & \square \end{bmatrix} = \det \begin{bmatrix} 1 & 4 \\ -1 & 9 \end{bmatrix} = (9 - (-4)) = 13$$

$$A_{ij} = (-1)^{2+3}(13) = -13$$

9–14 ■ Evaluate the minor and cofactor using the matrix A.

$$A = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ -3 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

10. M_{33}, A_{33}

11. M_{12}, A_{12}

Determinant of a Square Matrix:

If A is a n × n matrix, then the determinant of A is obtained by multiplying each element of the first row by its cofactors, and then adding the results.

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{bmatrix}$$

15-22 Find the determinant of the matrix. Determine whether the matrix has an inverse, but don't calculate the inverse.

16. $\begin{bmatrix} 0 & -1 & 0 \\ 2 & 6 & 4 \\ 1 & 0 & 3 \end{bmatrix}$

This is called expanding the determinant by the first row. We can expand the determinant by any row or column in the same way and obtain the same result in each case.

20. $\begin{bmatrix} 1 & 2 & 5 \\ -2 & -3 & 2 \\ 3 & 5 & 3 \end{bmatrix}$

23-26 Evaluate the determinant, using row or column operations whenever possible to simplify your work.

26. $\begin{vmatrix} 2 & -1 & 6 & 4 \\ 7 & 2 & -2 & 5 \\ 4 & -2 & 10 & 8 \\ 6 & 1 & 1 & 4 \end{vmatrix}$

Cramer's Rule for a 2 X 2 System

29-44 Use Cramer's Rule to solve the system.

30. $\begin{cases} 6x + 12y = 33 \\ 4x + 7y = 20 \end{cases}$

31. $\begin{cases} x - 6y = 3 \\ 3x + 2y = 1 \end{cases}$

Cramer's Rule for a 3 X 3 System

29-44 Use Cramer's Rule to solve the system.

36. $\begin{cases} 5x - 3y + z = 6 \\ 4y - 6z = 22 \\ 7x + 10y = -13 \end{cases}$

45-46 Evaluate the determinants

46. $\begin{vmatrix} a & a & a & a & a \\ 0 & a & a & a & a \\ 0 & 0 & a & a & a \\ 0 & 0 & 0 & a & a \\ 0 & 0 & 0 & 0 & a \end{vmatrix}$

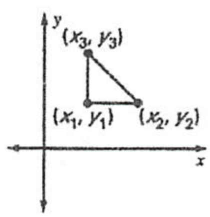
47-50 Solve for x.

49. $\begin{vmatrix} 1 & 0 & x \\ x^2 & 1 & 0 \\ x & 0 & 1 \end{vmatrix} = 0$

AREA OF A TRIANGLE

The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is given by

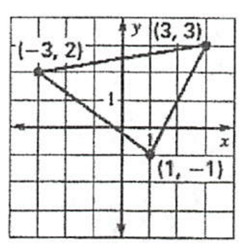
$$\text{Area} = \pm \frac{}{\rule{1em}{0.4pt}} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$



where the symbol ± indicates that the appropriate sign should be chosen to yield a ________ value.

Example 2 *The area of a triangle*

Find the area of the triangle shown.



$$\text{Area} = \pm\frac{1}{2} \begin{vmatrix} & & 1 \\ & & 1 \\ & & 1 \end{vmatrix}$$

____

$$= \pm\frac{1}{2}[\quad\quad\quad\quad\quad\quad\quad] = \_\_\_\_$$

Notes 7.4 Determinants and Cramer's Rule

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Determinant of a 2 X 2 Matrix

1-8 Find the determinant of the matrix, if it exists.

4. $\begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix}$ $4 - 3 = 1$

$-2(-2) - 3(1)$

6. $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ no determinant

The **minor** M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A.

The ***cofactor*** A_{ij} of the element a_{ij} is

$$A_{ij} = (-1)^{i+j} M_{ij}$$

To illustrate these definitions, consider the following 3 by 3 matrix,

$$\begin{bmatrix} 1 & 4 & 7 \\ 3 & 0 & 5 \\ -1 & 9 & 11 \end{bmatrix} \qquad M_{2,3} = \det \begin{bmatrix} 1 & 4 & \square \\ \square & \square & \square \\ -1 & 9 & \square \end{bmatrix} = \det \begin{bmatrix} 1 & 4 \\ -1 & 9 \end{bmatrix} = (9 - (-4)) = 13$$

$$A_{ij} = (-1)^{2+3}(13) = -13$$

9–14 ▪ Evaluate the minor and cofactor using the matrix A.

$$A = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ -3 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

$M_{12} = \det \begin{vmatrix} -3 & 2 \\ 0 & 4 \end{vmatrix} = -12 - 0 = -12$

10. M_{33}, A_{33}

$M_{33} = \det \begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix} = 5 - 0 = 5$

$A_{33} = (-1)^{3+3} \cdot 5 = \boxed{5}$

11. M_{12}, A_{12}

$A_{12} = (-1)^{1+2}(-12) = \boxed{12}$

Determinant of a Square Matrix:

If A is a n × n matrix, then the determinant of A is obtained by multiplying each element of the first row by its cofactors, and then adding the results.

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{bmatrix}$$

$2\begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} - 3\begin{vmatrix} 0 & 4 \\ -2 & 6 \end{vmatrix} + -1\begin{vmatrix} 0 & 2 \\ -2 & 5 \end{vmatrix}$

$= 2(-8) - 3(8) - 1(4)$

$= -44$

15-22 Find the determinant of the matrix. Determine whether the matrix has an inverse, but don't calculate the inverse.

16. $\begin{bmatrix} 0 & -1 & 0 \\ 2 & 6 & 4 \\ 1 & 0 & 3 \end{bmatrix}$

$$0\begin{vmatrix} 6 & 4 \\ 0 & 3 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} - 0\begin{vmatrix} 2 & 6 \\ 1 & 0 \end{vmatrix}$$

$$1(6-4)$$

$$= \boxed{2}$$

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This is called expanding the determinant by the first row. We can expand the determinant by any row or column in the same way and obtain the same result in each case.

20. $\begin{bmatrix} 1 & 2 & 5 \\ -2 & -3 & 2 \\ 3 & 5 & 3 \end{bmatrix}$

$$1\begin{vmatrix} -3 & 2 \\ 5 & 3 \end{vmatrix} - (-2)\begin{vmatrix} 2 & 5 \\ 5 & 3 \end{vmatrix} + 3\begin{vmatrix} 2 & 5 \\ -3 & 2 \end{vmatrix}$$

$$-9-10 \quad + 2(6-25) + 3(4+15)$$

$$-19 \quad - 38 \quad + 57 = 0$$

no inverse

23-26 Evaluate the determinant,
using row or column operations whenever possible to simplify your work.

26. $\left(\begin{vmatrix} 2 & -1 & 6 & 4 \\ 7 & 2 & -2 & 5 \\ 4 & -2 & 10 & 8 \\ 6 & 1 & 1 & 4 \end{vmatrix}\right) \cancel{x}\cdot 2 \quad \begin{vmatrix} 2 & -1 & 6 & 4 \\ 11 & 0 & 10 & 13 \\ 0 & 0 & -2 & 0 \\ 8 & 0 & 7 & 8 \end{vmatrix}$

$$-(-1)\begin{vmatrix} 11 & 10 & 13 \\ 0 & -2 & 0 \\ 8 & 7 & 8 \end{vmatrix} - 0\begin{vmatrix} \end{vmatrix} + 0\begin{vmatrix} \end{vmatrix} - 0\begin{vmatrix} \end{vmatrix}$$

$$-(-1)\left[11\begin{vmatrix} -2 & 0 \\ 7 & 8 \end{vmatrix} - 10\begin{vmatrix} 0 & 0 \\ 8 & 8 \end{vmatrix} + 13\begin{vmatrix} 0 & -2 \\ 8 & 7 \end{vmatrix}\right]$$

$$+1\left(11(-16) - 10(0) + 13(16)\right) = \boxed{32}$$

Cramer's Rule for a 2 X 2 System A is the coefficient matrix $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$ax+by=e$
$cx+dy=f$

$x = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{|A|}$ $y = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{|A|}$ $|A| \neq 0$

29-44 Use Cramer's Rule to solve the system.

30. $\begin{cases} 6x + 12y = 33 \\ 4x + 7y = 20 \end{cases}$ $A = \begin{vmatrix} 6 & 12 \\ 4 & 7 \end{vmatrix} = 42-48 = -6$

$x = \dfrac{\begin{vmatrix} 33 & 12 \\ 20 & 7 \end{vmatrix}}{-6} = \dfrac{-9}{-6} = \dfrac{3}{2}$

$y = \dfrac{\begin{vmatrix} 6 & 33 \\ 4 & 20 \end{vmatrix}}{-6} = \dfrac{-12}{-6} = 2$

31. $\begin{cases} x - 6y = 3 \\ 3x + 2y = 1 \end{cases}$ $A = \begin{vmatrix} 1 & -6 \\ 3 & 2 \end{vmatrix} = 20$

$x = \dfrac{\begin{vmatrix} 3 & -6 \\ 1 & 2 \end{vmatrix}}{20} = \dfrac{12}{20} = \dfrac{3}{5}$

$y = \dfrac{\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}}{20} = \dfrac{-8}{20} = \dfrac{-2}{5}$

$\boxed{\left(\frac{3}{5}, \frac{-2}{5}\right)}$

Cramer's Rule for a 3 X 3 System

$ax+by+cz=j$
$dx+ey+fz=k$
$gx+hy+iz=l$

$A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ $A \neq 0$

$x = \dfrac{\begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix}}{|A|}$ $y = \dfrac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{|A|}$ $z = \dfrac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}}{|A|}$

29-44 Use Cramer's Rule to solve the system.

36. $\begin{cases} 5x - 3y + z = 6 \\ 4y - 6z = 22 \\ 7x + 10y = -13 \end{cases}$

$A = \begin{vmatrix} 5 & -3 & 1 \\ 0 & 4 & -6 \\ 7 & 10 & 0 \end{vmatrix} 6 = \begin{vmatrix} 5 & -3 & 1 \\ 30 & -14 & 0 \\ 7 & 10 & 0 \end{vmatrix} = 1 \begin{vmatrix} 30 & -14 \\ 7 & 10 \end{vmatrix} = \boxed{398}$

$x = \dfrac{\begin{vmatrix} 6 & -3 & 1 \\ 22 & 4 & -6 \\ -13 & 10 & 0 \end{vmatrix} 6}{398} = \dfrac{\begin{vmatrix} 6 & -3 & 1 \\ 58 & -14 & 0 \\ -13 & 10 & 0 \end{vmatrix}}{398} = \dfrac{1\begin{vmatrix} 58 & -14 \\ -13 & 10 \end{vmatrix}}{398} = \dfrac{398}{398} = 1$

$y = \dfrac{\begin{vmatrix} 5 & 6 & 1 \\ 0 & 22 & -6 \\ 7 & -13 & 0 \end{vmatrix} 6}{398} = \dfrac{\begin{vmatrix} 5 & 6 & 1 \\ 30 & 58 & 0 \\ 7 & -13 & 0 \end{vmatrix}}{398} = \dfrac{1\begin{vmatrix} 30 & 58 \\ 7 & -13 \end{vmatrix}}{398} = \dfrac{-796}{398} = -2$

$\boxed{(1, -2, -5)}$

$z = \dfrac{\begin{vmatrix} 5 & -3 & 6 \\ 0 & 4 & 22 \\ 7 & 10 & -13 \end{vmatrix}}{398} = 5\begin{vmatrix} 4 & 22 \\ 10 & -13 \end{vmatrix} + 7\begin{vmatrix} -3 & 6 \\ 4 & 22 \end{vmatrix} = 5(-272) + 7(-90) = \dfrac{-1990}{398}$

$= -5$

45-46 Evaluate the determinants

46. $\begin{vmatrix} a & a & a & a & a \\ 0 & a & a & a & a \\ 0 & 0 & a & a & a \\ 0 & 0 & 0 & a & a \\ 0 & 0 & 0 & 0 & a \end{vmatrix} = a\begin{vmatrix} a & a & a & a \\ 0 & a & a & a \\ 0 & 0 & a & a \\ 0 & 0 & 0 & a \end{vmatrix} = a\left(a\begin{vmatrix} a & a & a \\ 0 & a & a \\ 0 & 0 & a \end{vmatrix}\right)$

$= a\left(a\left(a\begin{vmatrix} a & a \\ 0 & a \end{vmatrix}\right)\right) = a \cdot a \cdot a \cdot a^2$

$\boxed{= a^5}$

47-50 Solve for x.

49. $\begin{vmatrix} 1 & 0 & x \\ x^2 & 1 & 0 \\ x & 0 & 1 \end{vmatrix} = 0$

$1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} x^2 & 0 \\ x & 1 \end{vmatrix} + x\begin{vmatrix} x^2 & 1 \\ x & 0 \end{vmatrix}$

$1(1) + x(-x)$

$1 - x^2 = 0$

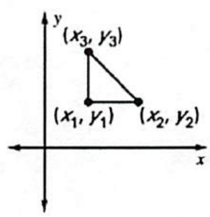
$1 = x^2$

$\boxed{x = \pm 1}$

AREA OF A TRIANGLE

The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is given by

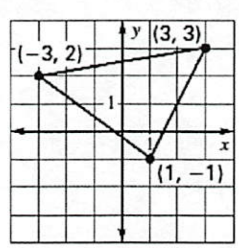
$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$



where the symbol ± indicates that the appropriate sign should be chosen to yield a positive value.

Example 2 *The area of a triangle*

Find the area of the triangle shown.



$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} -3 & 2 & 1 \\ 3 & 3 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$-3\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & 3 \\ 1 & -1 \end{vmatrix}$

$= \pm\frac{1}{2}[-3(4) - 2(2) + 1(-6)] = 11$